Coherent states for Hamiltonians generated by supersymmetry

2007 J. Phys. A: Math. Theor. 40 6491

(http://iopscience.iop.org/1751-8121/40/24/015)

J. Phys. A: Math. Theor. **40** (2007) 6491–6511 doi:10.1088/1751-8113/40/24/015

Coherent states for Hamiltonians generated by supersymmetry

David J Fernández[1], **Véronique Hussin**[2] **and Oscar Rosas-Ortiz**[1]

[1] Departamento de Física, Cinvestav, AP 14-740, 07000 México D F, Mexico
[2] Département de Mathématiques et Centre de Recherches Mathématiques, Université de Montréal, C. P. 6128, Succ. Centre-Ville, Montréal, QC H3C 3J7, Canada

Received 15 January 2007, in final form 29 April 2007
Published 30 May 2007
Online at stacks.iop.org/JPhysA/40/6491

Abstract
Coherent states are derived for one-dimensional systems generated by supersymmetry from an initial Hamiltonian with a purely discrete spectrum for which the levels depend analytically on their subindex. It is shown that the algebra of the initial system is inherited by its SUSY partners in the subspace associated with the isospectral part or the spectrum. The technique is applied to the harmonic oscillator, infinite well and trigonometric Pöschl–Teller potentials.

PACS numbers: 11.30.Pb, 03.65.Ge, 03.65.Fd, 02.30.Gp

1. Introduction

The great interest in the study of coherent states (CS) stems from the beautiful properties that the so-called standard ones have, which are a natural consequence of the huge symmetry supplied by the Heisenberg–Weyl algebra ruling the harmonic oscillator. Indeed, these characteristics suggested Glauber to model light by means of standard coherent states [1], which was a breakthrough in the development of quantum optics, one of most successful branches of the physics of the twentieth century (see, e.g., [2–7]).

Among the several definitions available in the literature for general systems, algebraically the most important ones are those which define the CS either as eigenstates of annihilation operators or as resulting of a 'displacement' operator acting onto a certain extremal state. In order to derive the CS following the first definition, one has to identify the appropriate algebra ruling the system Hamiltonian, and then to find the annihilation and creation operators suitable to perform the construction. Since typically the resulting algebra is not linear, it is usual to call them nonlinear coherent states [8–16].

For Hamiltonians H_k generated by supersymmetric quantum mechanics (SUSY QM) [17–28], the CS analysis has been focussed mainly on the SUSY partners of the harmonic oscillator [29–34] (see, however, [35, 36]). The key ingredient in the approach introduced in [29, 34] is to construct a *natural* pair of annihilation and creation operators of H_k simply as products of intertwining and standard annihilation and creation operators. An important conclusion of these works was that the natural algebra ruling the SUSY partner Hamiltonians of the oscillator is a polynomial deformation of the Heisenberg–Weyl algebra.

For the SUSY partners of a general initial potential, an appropriate algebraic treatment of the corresponding Hamiltonian H_0, ensuring a right identification of the annihilation and creation operators, has not been realized. However, for a set of one-dimensional Hamiltonians with a purely discrete spectrum for which the levels depend analytically on their index, an *intrinsic* algebra has been identified recently, allowing us to calculate in a simple way the corresponding CS [37]. Let us note that this intrinsic algebra is in general nonlinear. One of the results of the present paper is to show that such algebraic structures can be linearized: one can associate with those systems the Heisenberg–Weyl algebra. Consequently, an additional set of CS will be constructed, their explicit expressions containing small variations from the standard harmonic oscillator CS.

It is remarkable that [37] as well draws attention to the main subject of this paper, namely, the CS analysis for the SUSY partners of arbitrary potentials in the spirit of [29, 34]. In this context several novel results will be found, e.g., we will show that the nonlinear and linear algebras of H_0 are inherited by its SUSY partners H_k in the subspace associated with the isospectral part of the spectrum. In addition, we will find a *natural* algebra for which the generators are products of annihilation and creation operators of H_0 times the intertwiners of H_0 and H_k, thus generalizing the previous results for the harmonic oscillator [29, 34]. The corresponding CS will be built up for the several algebras of H_k we are going to study. Our procedure will be illustrated with the harmonic oscillator, infinite well and trigonometric Pöschl–Teller potentials. The results for the SUSY partners of the infinite well and trigonometric Pöschl–Teller potentials, as far as we know, are new.

Let us observe that for specific potentials, such as trigonometric Pöschl–Teller, Morse and others, there are alternative methods of construction of CS which employ the symmetry of the differential equations related to H_0 (see, e.g., [38]). However, to implement the SUSY transformations departing from such treatments seems involved, as compared with the technique which will be presented in this paper (based on [37]).

In the next section the initial Hamiltonian we deal with as well as its related algebras will be studied. The CS analysis for the several algebras of H_0 is the subject of section 3. A brief overview of SUSY QM as a technique for generating solvable potentials from a given initial one will be presented in section 4. In section 5, a pair of nonlinear algebras ruling the SUSY partner potentials will be discussed, while in section 6 we will explore the corresponding linear structure. The CS construction for the several algebras associated with the SUSY partner potentials will be performed in section 7. In section 8 our general results will be illustrated with some examples. Finally, in section 9 we close the paper with our conclusions.

2. Algebraic structures of the initial Hamiltonian H_0

Let us suppose that the initial system is described by a Hermitian Schrödinger Hamiltonian

$$H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(x), \tag{2.1}$$

whose eigenvectors and eigenvalues satisfy

$$H_0|\psi_n\rangle = E_n|\psi_n\rangle, \qquad E_0 < E_1 < E_2 < \cdots. \tag{2.2}$$

We assume that there is an analytic dependence, defined by a certain function $E(n)$, of the eigenvalues with the index labelling them, namely

$$E_n \equiv E(n), \tag{2.3}$$

and the eigenvectors satisfy the standard orthonormality and completeness relationships

$$\langle\psi_m|\psi_n\rangle = \delta_{mn}, \qquad \sum_{m=0}^{\infty} |\psi_m\rangle\langle\psi_m| = 1, \tag{2.4}$$

where the symbol 1 in any operator expression of this paper represents the identity operator. There will be different forms of $E(n)$ according to the system under study, for instance, for the harmonic oscillator it will be a linear function of n, for an infinite square well it will be quadratic, etc. This function defines an *intrinsic* algebra which will next be discussed.

2.1. Intrinsic nonlinear algebra of H_0

Let us define a pair of annihilation and creation operators $a_0^{\pm}$ by

$$a_0^-|\psi_n\rangle = r_{\mathcal{I}}(n)|\psi_{n-1}\rangle, \qquad a_0^+|\psi_n\rangle = \bar{r}_{\mathcal{I}}(n+1)|\psi_{n+1}\rangle, \tag{2.5}$$

$$r_{\mathcal{I}}(n) = \mathrm{e}^{\mathrm{i}\alpha(E_n - E_{n-1})}\sqrt{E_n - E_0}, \qquad \alpha \in \mathbb{R}, \tag{2.6}$$

such that their product becomes

$$a_0^+a_0^- = H_0 - E_0. \tag{2.7}$$

The number operator N_0 is now introduced with the properties

$$N_0|\psi_n\rangle = n|\psi_n\rangle, \qquad \left[N_0, a_0^{\pm}\right] = \pm a_0^{\pm}. \tag{2.8}$$

As a consequence, two equations which will be widely used along this work are obtained:

$$a_0^{\pm} g(N_0) = g(N_0 \mp 1)a_0^{\pm}, \tag{2.9}$$

$g(x)$ being a real arbitrary non-singular function for $x \in \mathbb{Z}^+$. Combining equations (2.2), (2.5)–(2.8), it turns out that the *intrinsic algebra* of the system is characterized by the relationships

$$H_0 = E(N_0), \qquad a_0^+a_0^- = E(N_0) - E_0, \qquad a_0^-a_0^+ = E(N_0+1) - E_0, \tag{2.10}$$

$$\left[a_0^-, a_0^+\right] = E(N_0+1) - E(N_0) \equiv f(N_0), \tag{2.11}$$

$$\left[H_0, a_0^{\pm}\right] = \pm a_0^{\pm} f(N_0 - 1/2 \pm 1/2) = \pm f(N_0 - 1/2 \mp 1/2)a_0^{\pm}. \tag{2.12}$$

We will see below that this is not the only algebra of H_0 which can be defined.

Let us note that we can express $a_0^{\pm}$ in the form

$$a_0^- = r_{\mathcal{I}}(N_0+1)\sum_{m=0}^{\infty}|\psi_m\rangle\langle\psi_{m+1}|, \qquad a_0^+ = \bar{r}_{\mathcal{I}}(N_0)\sum_{m=0}^{\infty}|\psi_{m+1}\rangle\langle\psi_m|, \tag{2.13}$$

where each term in both summations is a Hubbard operator [39–41]. Hence, throughout this paper we will call these decompositions *Hubbard representations*.

2.2. *Linear algebra of H_0*

The *intrinsic* algebra (2.8), (2.10)–(2.12) admits a linearizing procedure, i.e., one can build up new annihilation and creation operators satisfying the standard oscillator algebra [29, 34]. Let us construct them in the form

$$a_{0_\mathcal{L}}^- = b(N_0)a_0^- = a_0^- b(N_0 - 1), \qquad a_{0_\mathcal{L}}^+ = a_0^+ b(N_0) = b(N_0 - 1)a_0^+, \tag{2.14}$$

$b(x)$ being a real non-singular function for $x \in \mathbb{Z}^+$ to be determined. Suppose that the action of $a_{0_\mathcal{L}}^\pm$ onto the eigenvectors of H_0, up to the same phase factors as in (2.5)–(2.6), is equal to the oscillator one, namely

$$a_{0_\mathcal{L}}^- |\psi_n\rangle = r_\mathcal{L}(n)|\psi_{n-1}\rangle, \qquad a_{0_\mathcal{L}}^+ |\psi_n\rangle = \bar{r}_\mathcal{L}(n+1)|\psi_{n+1}\rangle, \tag{2.15}$$

$$r_\mathcal{L}(n) = \mathrm{e}^{\mathrm{i}\alpha f(n-1)}\sqrt{n}. \tag{2.16}$$

On the other hand, the expressions for $a_{0_\mathcal{L}}^\pm$ given in (2.14) and the use of (2.5) lead to

$$a_{0_\mathcal{L}}^- |\psi_n\rangle = b(n-1)r_\mathcal{I}(n)|\psi_{n-1}\rangle, \qquad a_{0_\mathcal{L}}^+ |\psi_n\rangle = b(n)\bar{r}_\mathcal{I}(n+1)|\psi_{n+1}\rangle. \tag{2.17}$$

By comparing (2.15) with (2.17) we get

$$b(n) = \frac{\bar{r}_\mathcal{L}(n+1)}{\bar{r}_\mathcal{I}(n+1)} = \frac{r_\mathcal{L}(n+1)}{r_\mathcal{I}(n+1)} = \sqrt{\frac{n+1}{E(n+1) - E_0}}. \tag{2.18}$$

Making use of (2.13)–(2.14), (2.18), the Hubbard representation of $a_{0_\mathcal{L}}^\pm$ is obtained,

$$a_{0_\mathcal{L}}^- = r_\mathcal{L}(N_0 + 1)\sum_{m=0}^\infty |\psi_m\rangle\langle\psi_{m+1}|, \qquad a_{0_\mathcal{L}}^+ = \bar{r}_\mathcal{L}(N_0)\sum_{m=0}^\infty |\psi_{m+1}\rangle\langle\psi_m|, \tag{2.19}$$

which, up to the exponential factors of $r_\mathcal{L}$, is equal to the oscillator one. Let us note that, as a consequence of (2.9), we get $a_{0_\mathcal{L}}^\pm g(N_0) = g(N_0 \mp 1)a_{0_\mathcal{L}}^\pm$. Thus, the set $\{N_0, a_{0_\mathcal{L}}^-, a_{0_\mathcal{L}}^+\}$ satisfies the oscillator algebra:

$$\left[N_0, a_{0_\mathcal{L}}^\pm\right] = \pm a_{0_\mathcal{L}}^\pm, \qquad a_{0_\mathcal{L}}^+ a_{0_\mathcal{L}}^- = N_0, \qquad a_{0_\mathcal{L}}^- a_{0_\mathcal{L}}^+ = N_0 + 1, \qquad \left[a_{0_\mathcal{L}}^-, a_{0_\mathcal{L}}^+\right] = 1. \tag{2.20}$$

However, the commutator of H_0 with $a_{0_\mathcal{L}}^\pm$ remains the same as for $a_0^\pm$ (see equation (2.12)).

2.3. *General deformation of the intrinsic algebra of H_0*

Since it will be used later, it is worthwhile to mention that the previous linearization is a particular case of a general deformation of the intrinsic algebra defined by equation (2.8), (2.10)–(2.12) for N_0, a_0^-, a_0^+. In this procedure, new annihilation and creation operators $a^- = \beta(N_0)a_0^-$, $a^+ = a_0^+\beta(N_0)$, are constructed such that

$$[N_0, a^\pm] = \pm a^\pm, \qquad a^+a^- = \widetilde{E}(N_0), \qquad a^-a^+ = \widetilde{E}(N_0 + 1), \tag{2.21}$$

$$[a^-, a^+] = \widetilde{E}(N_0 + 1) - \widetilde{E}(N_0) = \widetilde{f}(N_0), \tag{2.22}$$

where $\widetilde{E}(N_0)$ and $\widetilde{E}(N_0 + 1)$ are positive definite operators and $\beta(x)$ is a real non-singular function for $x \in \mathbb{Z}^+$ to be adjusted according to the chosen $\widetilde{E}(N_0)$. It is clear that different choices of $\widetilde{E}(N_0)$ lead to different deformations:

$$\widetilde{E}(N_0) = \beta^2(N_0 - 1)[E(N_0) - E_0] \quad \Rightarrow \quad \beta(N_0) = \sqrt{\frac{\widetilde{E}(N_0 + 1)}{E(N_0 + 1) - E_0}}. \tag{2.23}$$

In particular, in the previous section we were interested in a deformation simplifying maximally the original algebra. It can be here recovered by the choice $\widetilde{E}(N_0) = N_0$, and by using (2.14), (2.18), (2.23), it turns out that $\beta(x) = b(x)$, $a^\pm = a_{0_\mathcal{L}}^\pm$, $\widetilde{f}(N_0) = 1$.

3. Coherent states of H_0

Once some algebras ruling our system have been identified, let us look for the associated CS. We will derive them as eigenstates of the several annihilation operators defined previously.

3.1. Intrinsic nonlinear coherent states of H_0

In the first place, let us analyse the CS $|z,\alpha\rangle_0$ which are eigenstates of the annihilation operator of the intrinsic algebra:

$$a_0^-|z,\alpha\rangle_0 = z|z,\alpha\rangle_0, \qquad z \in \mathbb{C}. \tag{3.1}$$

By expanding $|z,\alpha\rangle_0$ in the basis of eigenstates of H_0 and following the standard procedure to determine the expansion coefficients, it turns out that

$$|z,\alpha\rangle_0 = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \mathrm{e}^{-\mathrm{i}\alpha(E_m-E_0)}\frac{z^m}{\sqrt{\rho_m}}|\psi_m\rangle, \tag{3.2}$$

$$\rho_m = \begin{cases} 1 & \text{if} \quad m = 0, \\ (E_m - E_0)\cdots(E_1 - E_0) & \text{if} \quad m > 0. \end{cases} \tag{3.3}$$

It is now important to seek if the intrinsic nonlinear CS (3.2) form a complete set, i.e., if they satisfy

$$\int |z,\alpha\rangle_{00}\langle z,\alpha|\,\mathrm{d}\mu(z) = 1. \tag{3.4}$$

Let us express the positive definite measure $\mathrm{d}\mu(z)$ in the form

$$\mathrm{d}\mu(z) = \frac{1}{\pi}\left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)\rho(|z|^2)\,\mathrm{d}^2z, \tag{3.5}$$

$\rho(y)$ being a function to be determined. Working in polar coordinates and making the change of variable $y = |z|^2$, it is straightforward to show that $\rho(y)$ must satisfy

$$\int_0^{\infty} y^m\rho(y)\,\mathrm{d}y = \rho_m, \qquad m = 0, 1, \ldots. \tag{3.6}$$

The moment problem (3.6), in which we look for a positive definite function $\rho(y)$ with the given mth order moments ρ_m, often arises in the literature when a completeness relationship of kind (3.4) is to be proven [29, 34, 42–44]. The generic answer is nowadays known: $\rho(y)$ is the inverse Mellin transform of ρ_m [34]. However, for each particular system this calculation has to be performed explicitly, which is not always easy (see, e.g., [29]).

Expression (3.4) guarantees that any state of the system can be expanded in terms of CS. In particular, this can be done for an arbitrary CS $|z',\alpha\rangle_0$:

$$|z',\alpha\rangle_0 = \int |z,\alpha\rangle_{00}\langle z,\alpha|z',\alpha\rangle_0\,\mathrm{d}\mu(z), \tag{3.7}$$

where the reproducing kernel ${}_0\langle z,\alpha|z',\alpha\rangle_0$ is expressed as

$${}_0\langle z,\alpha|z',\alpha\rangle_0 = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}}\left(\sum_{m=0}^{\infty}\frac{|z'|^{2m}}{\rho_m}\right)^{-\frac{1}{2}}\left(\sum_{m=0}^{\infty}\frac{(\bar{z}z')^m}{\rho_m}\right). \tag{3.8}$$

Let us note that the eigenvalue $z = 0$ of a_0^- is non-degenerated since

$$|z = 0,\alpha\rangle_0 = |\psi_0\rangle. \tag{3.9}$$

Another important property of the intrinsic nonlinear CS $|z,\alpha\rangle_0$, which is due to the phase choice of equation (2.5)–(2.6), is that they evolve coherently in time:

$$U_0(t)|z,\alpha\rangle_0 = \mathrm{e}^{-\mathrm{i}tE_0}|z,\alpha+t\rangle_0, \tag{3.10}$$

$U_0(t) = \exp(-\mathrm{i}tH_0)$ being the evolution operator associated with H_0.

3.2. Linear coherent states of H_0

Let us study the CS which are eigenstates of the linear annihilation operator of H_0:

$$a_{0_\mathcal{L}}^-|z,\alpha\rangle_{0_\mathcal{L}} = z|z,\alpha\rangle_{0_\mathcal{L}}, \qquad z\in\mathbb{C}. \tag{3.11}$$

Hence

$$|z,\alpha\rangle_{0_\mathcal{L}} = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{m=0}^{\infty}\mathrm{e}^{-\mathrm{i}\alpha(E_m-E_0)}\frac{z^m}{\sqrt{m!}}|\psi_m\rangle. \tag{3.12}$$

Up to the phases involving α, they have the form of the standard harmonic oscillator CS.

Contrasting with the difficulty to find a positive definite measure ensuring the completeness of the nonlinear CS (3.2), now the problem is already solved:

$$\frac{1}{\pi}\int |z,\alpha\rangle_{0_\mathcal{L}}\,{}_{0_\mathcal{L}}\langle z,\alpha|\,\mathrm{d}^2z = 1, \tag{3.13}$$

i.e., the measure is the standard one, d^2z/π. Thus, an arbitrary linear CS $|z',\alpha\rangle_{0_\mathcal{L}}$ admits a non-trivial decomposition in terms of $|z,\alpha\rangle_{0_\mathcal{L}}$:

$$|z',\alpha\rangle_{0_\mathcal{L}} = \frac{1}{\pi}\int |z,\alpha\rangle_{0_\mathcal{L}}\,{}_{0_\mathcal{L}}\langle z,\alpha|z',\alpha\rangle_{0_\mathcal{L}}\mathrm{d}^2z, \tag{3.14}$$

where the reproducing kernel is equal to the oscillator one:

$${}_{0_\mathcal{L}}\langle z,\alpha|z',\alpha\rangle_{0_\mathcal{L}} = \exp\left(-\frac{|z|^2}{2} + \bar{z}z' - \frac{|z'|^2}{2}\right). \tag{3.15}$$

The only eigenstate of H_0 which is as well a linear CS (3.12) is again the ground state:

$$|z=0,\alpha\rangle_{0_\mathcal{L}} = |\psi_0\rangle. \tag{3.16}$$

Since $\left[a_{0_\mathcal{L}}^-, a_{0_\mathcal{L}}^+\right] = 1$, the linear CS also result from acting a 'displacement' operator onto $|\psi_0\rangle$:

$$|z,\alpha\rangle_{0_\mathcal{L}} = D_\mathcal{L}(z)|\psi_0\rangle = \exp\left(za_{0_\mathcal{L}}^+ - \bar{z}a_{0_\mathcal{L}}^-\right)|\psi_0\rangle. \tag{3.17}$$

4. The SUSY partner Hamiltonians H_k

Let us discuss in the first place some generalities of the SUSY partner Hamiltonians H_k,

$$H_k = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_k(x), \tag{4.1}$$

generated from H_0 through a kth order differential intertwining operator B_k^+ [34],

$$H_kB_k^+ = B_k^+H_0 \quad\Leftrightarrow\quad H_0B_k = B_kH_k. \tag{4.2}$$

The potential $V_k(x)$ reads

$$V_k(x) = V_0(x) - \sum_{i=1}^{k}\alpha_i'(x,\epsilon_i), \tag{4.3}$$

where, in case that the k factorization energies ϵ_i, $i = 1, \ldots, k$ are all different, $\alpha_i(x, \epsilon_i)$ is obtained from a recursive (Bäcklund) formula,

$$\alpha_i(x, \epsilon_i) = -\alpha_{i-1}(x, \epsilon_{i-1}) - \frac{2(\epsilon_i - \epsilon_{i-1})}{\alpha_{i-1}(x, \epsilon_i) - \alpha_{i-1}(x, \epsilon_{i-1})}, \qquad i = 2, \ldots k, \tag{4.4}$$

and $\alpha_1(x, \epsilon_i)$ are solutions of the following Riccati equation:

$$\alpha_1'(x, \epsilon_i) + \alpha_1^2(x, \epsilon_i) = 2[V_0(x) - \epsilon_i], \qquad i = 1, \ldots, k. \tag{4.5}$$

This is equivalent to the initial stationary Schrödinger equation for the factorization energies ϵ_i, as can be seen from the change $\alpha_1(x, \epsilon_i) = u_i'(x)/u_i(x)$:

$$-\tfrac{1}{2}u_i'' + V_0(x)u_i = \epsilon_i u_i. \tag{4.6}$$

In terms of the transformation functions u_i, the new potential in (4.3) becomes

$$V_k(x) = V_0(x) - \{\ln[W(u_1, \ldots, u_k)]\}'', \tag{4.7}$$

with $W(u_1, \ldots, u_k)$ being the Wronskian of the involved solutions of (4.6). It is worthwhile to note that, in order to obtain nontrivial results when two (or more) factorization energies coincide, the confluent limit of the previous formulae has to be used [45, 46]. It is important also to write down the relevant factorizations for the SUSY QM of kth order:

$$B_k^+ B_k = \prod_{i=1}^{k}(H_k - \epsilon_i), \qquad B_k B_k^+ = \prod_{i=1}^{k}(H_0 - \epsilon_i). \tag{4.8}$$

Let us now suppose that, as a result of the kth order intertwining technique, s of the states annihilated by B_k are as well physical eigenstates of H_k associated with the eigenvalues ϵ_i. By convenience, they will be specially denoted by $|\theta_{\epsilon_i}\rangle$, $B_k|\theta_{\epsilon_i}\rangle = 0$, $H_k|\theta_{\epsilon_i}\rangle = \epsilon_i|\theta_{\epsilon_i}\rangle$, $i = 1, \ldots, s$, $s \leqslant k$. However, we assume that the procedure creates just q additional levels with respect to $\mathrm{Sp}(H_0)$, but without deleting any of the original levels of H_0, i.e.,

$$\mathrm{Sp}(H_k) = \{\epsilon_1, \ldots, \epsilon_q, E_0, E_1, \ldots\}, \qquad q \leqslant s. \tag{4.9}$$

This means that $p \equiv s - q$ factorization energies ϵ_{q+j} coincide with p energy levels E_{m_j} of H_0, i.e., $\epsilon_{q+j} = E_{m_j}$, $j = 1, \ldots, p$, $m_j < m_{j+1}$, and thus $B_k^+|\psi_{m_j}\rangle = 0$. The eigenstates $|\theta_n\rangle$ of H_k which are associated with the remaining energies E_n, $n \neq m_j$, are obtained from the initial ones $|\psi_n\rangle$ and vice versa through the intertwining operators B_k^+ and B_k, namely

$$|\theta_n\rangle = \frac{B_k^+|\psi_n\rangle}{\sqrt{\prod_{i=1}^{k}(E_n - \epsilon_i)}}, \qquad |\psi_n\rangle = \frac{B_k|\theta_n\rangle}{\sqrt{\prod_{i=1}^{k}(E_n - \epsilon_i)}}. \tag{4.10}$$

It is convenient now to extend the definition of $|\theta_n\rangle$ for $n = m_j$ in the way,

$$|\theta_{m_j}\rangle \equiv |\theta_{\epsilon_{q+j}}\rangle, \qquad j = 1, \ldots, p. \tag{4.11}$$

Summarizing all this information, the eigenstates $|\theta_{\epsilon_i}\rangle$, $|\theta_n\rangle$ of H_k obey

$$H_k|\theta_n\rangle = E_n|\theta_n\rangle, \qquad H_k|\theta_{\epsilon_i}\rangle = \epsilon_i|\theta_{\epsilon_i}\rangle, \tag{4.12}$$

$$\langle\theta_{\epsilon_i}|\theta_n\rangle = 0, \qquad \langle\theta_m|\theta_n\rangle = \delta_{mn}, \qquad \langle\theta_{\epsilon_i}|\theta_{\epsilon_j}\rangle = \delta_{ij}, \tag{4.13}$$

$$\sum_{l=1}^{s}|\theta_{\epsilon_l}\rangle\langle\theta_{\epsilon_l}| + \widetilde{\sum_{m}}|\theta_m\rangle\langle\theta_m| = \sum_{l=1}^{q}|\theta_{\epsilon_l}\rangle\langle\theta_{\epsilon_l}| + \sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_m| = 1, \tag{4.14}$$

where $n, m = 0, 1, \ldots, i, j = 1, \ldots, q$, $\widetilde{\sum}_m$ is the sum over $m = 0, 1, \ldots$ except by m_j, $j = 1, \ldots, p$, and the identity operator has been expanded in two alternative ways which will be useful later. Since the positions of the new levels ϵ_i, $i = 1, \ldots, q$, are arbitrary, one might think that some algebraic properties of H_0 are inherited by H_k on the subspace spanned by the $|\theta_n\rangle$, $n = 0, 1, \ldots$. Keeping this in mind, let us analyse some interesting algebras of the SUSY partner Hamiltonians H_k.

5. Nonlinear algebras of H_k

We define first a *number operator* N_k by its action onto the eigenstates of H_k:

$$N_k|\theta_n\rangle = n|\theta_n\rangle, \qquad N_k\left|\theta_{\epsilon_i}\right\rangle = 0, \qquad n = 0, 1, \ldots, i = 1, \ldots, q. \tag{5.1}$$

Note that this definition is more natural than a previous one, introduced as the 'generalized number operator' for the SUSY partners of the oscillator (cf equation (3.4) of [34]).

Let us study next two pairs of annihilation and creation operators of H_k (and N_k) as well as their corresponding nonlinear algebras.

5.1. Natural algebra of H_k

Here we will obtain annihilation and creation operators of H_k following a 3-step construction previously introduced for the SUSY partner Hamiltonians of the harmonic oscillator [29, 34, 47]. Thus, starting from the *intrinsic* operators $a_0^\pm$ of H_0 and the intertwining ones B_k, B_k^+ of (4.2), a pair of *natural* annihilation and creation operators $a_{k_\mathcal{N}}^\pm$ of H_k is built up:

$$a_{k_\mathcal{N}}^\pm = B_k^+ a_0^\pm B_k. \tag{5.2}$$

Since $B_k\left|\theta_{\epsilon_i}\right\rangle = 0$, $i = 1, \ldots, s$, one can find the action of $a_{k_\mathcal{N}}^\pm$ onto the basis of eigenvectors of H_k (and N_k) by using (2.5), (4.10):

$$a_{k_\mathcal{N}}^\pm\left|\theta_{\epsilon_i}\right\rangle = 0, \qquad i = 1, \ldots, q, \tag{5.3}$$

$$a_{k_\mathcal{N}}^-|\theta_n\rangle = r_\mathcal{N}(n)|\theta_{n-1}\rangle, \qquad a_{k_\mathcal{N}}^+|\theta_n\rangle = \bar{r}_\mathcal{N}(n+1)|\theta_{n+1}\rangle, \qquad n = 0, 1, \ldots \tag{5.4}$$

$$r_\mathcal{N}(n) = \left\{\prod_{i=1}^{k}[E(n) - \epsilon_i][E(n-1) - \epsilon_i]\right\}^{\frac{1}{2}} r_\mathcal{I}(n). \tag{5.5}$$

Note that $r_\mathcal{N}(m_j) = 0$, $j = 1, \ldots, p$, which is consistent with $B_k\left|\theta_{m_j}\right\rangle = a_{k_\mathcal{N}}^-\left|\theta_{m_j}\right\rangle = 0$. From these expressions one can find the Hubbard representation for $a_{k_\mathcal{N}}^\pm$:

$$a_{k_\mathcal{N}}^- = r_\mathcal{N}(N_k+1)\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_{m+1}|, \qquad a_{k_\mathcal{N}}^+ = \bar{r}_\mathcal{N}(N_k)\sum_{m=0}^{\infty}|\theta_{m+1}\rangle\langle\theta_m|. \tag{5.6}$$

Making use of $a_{k_\mathcal{N}}^\pm g(N_k) = g(N_k \mp 1)a_{k_\mathcal{N}}^\pm$ for an arbitrary regular function $g(x)$, $x \in \mathbb{Z}^+$, one can show that

$$\left[a_{k_\mathcal{N}}^-, a_{k_\mathcal{N}}^+\right] = [\bar{r}_\mathcal{N}(N_k+1)r_\mathcal{N}(N_k+1) - \bar{r}_\mathcal{N}(N_k)r_\mathcal{N}(N_k)]\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_m|. \tag{5.7}$$

5.2. Intrinsic algebra of H_k

It is interesting to observe that simpler annihilation and creation operators for H_k can be constructed, proceeding by analogy with (2.13). Thus, we define the *intrinsic* annihilation and creation operators $a_k^\pm$ of H_k as follows:

$$a_k^- = r_\mathcal{I}(N_k+1)\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_{m+1}|, \qquad a_k^+ = \bar{r}_\mathcal{I}(N_k)\sum_{m=0}^{\infty}|\theta_{m+1}\rangle\langle\theta_m|, \tag{5.8}$$

where $r_{\mathcal{I}}(n)$ is given in (2.6). It can be checked that $a_k^{\pm}|\theta_{\epsilon_i}\rangle = 0$, $i = 1, \ldots, q$, and

$$a_k^-|\theta_n\rangle = r_{\mathcal{I}}(n)|\theta_{n-1}\rangle, \qquad a_k^+|\theta_n\rangle = \bar{r}_{\mathcal{I}}(n+1)|\theta_{n+1}\rangle, \tag{5.9}$$

$$a_k^+a_k^-|\theta_n\rangle = (E_n - E_0)|\theta_n\rangle, \qquad a_k^-a_k^+|\theta_n\rangle = (E_{n+1} - E_0)|\theta_n\rangle. \tag{5.10}$$

Thus, the commutator between $a_k^{\pm}$ is similar to that for the intrinsic algebra of H_0 on the subspace spanned by $\{|\theta_n\rangle, n = 0, 1, \ldots\}$:

$$\left[a_k^-, a_k^+\right] = f(N_k)\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_m|. \tag{5.11}$$

We would like to seek next if there is any connection between the initial and final number operators N_0 and N_k. After some simple manipulations, it can be shown that

$$N_k = C_k^+N_0C_k + \sum_{j=1}^{p} m_j|\theta_{m_j}\rangle\langle\theta_{m_j}| \Leftrightarrow N_k\widetilde{\sum_m}|\theta_m\rangle\langle\theta_m| = C_k^+N_0C_k, \tag{5.12}$$

$$C_k = \frac{1}{\sqrt{\prod_{i=1}^{k}[E(N_0) - \epsilon_i]}}B_k, \qquad C_k^+ = \frac{1}{\sqrt{\prod_{i=1}^{k}[E(N_k) - \epsilon_i]}}B_k^+, \tag{5.13}$$

C_k, C_k^+ being *modified intertwining operators* inverse to each other when acting on the eigenstates of the isospectral part which are not used as seeds in the SUSY procedure, i.e.,

$$C_k|\theta_n\rangle = |\psi_n\rangle, \qquad C_k^+|\psi_n\rangle = |\theta_n\rangle, \qquad \mathbb{Z}^+ \ni n \neq m_j, \qquad j = 1, \ldots, p, \tag{5.14}$$

but in general they are not invertible in the full Hilbert space $\mathcal{L}^2(\mathbb{R})$ since $C_k|\theta_{\epsilon_i}\rangle = C_k|\theta_{m_j}\rangle = C_k^+|\psi_{m_j}\rangle = 0$, $i = 1, \ldots, q$, $j = 1, \ldots, p$. From these expressions one can check that

$$a_k^{\pm}\widetilde{\sum_m}|\theta_m\rangle\langle\theta_m| = C_k^+a_0^{\pm}C_k. \tag{5.15}$$

By using equations (5.14)–(5.15) one recovers (5.9). Moreover, it turns out that

$$a_k^+a_k^- = [E(N_k) - E_0] = [H_k - E_0]\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_m|. \tag{5.16}$$

The RHS of expressions (5.15) for the intrinsic operators $a_k^{\pm}$ consist of a 3-step action, similar to the natural ones $a_{k_{\mathcal{N}}}^{\pm}$ of (5.2). The difference is that the new intertwiners C_k, C_k^+ transform the states $|\theta_n\rangle \leftrightarrow |\psi_n\rangle, \mathbb{Z}^+ \ni n \neq m_j, j = 1, \ldots, p$, without changing the norm (compare (5.14) with (4.10)). This explains why the *intrinsic* algebra generated by $\{N_k, a_k^-, a_k^+\}$ is simpler than the *natural* one obtained from $\{N_k, a_{k_{\mathcal{N}}}^-, a_{k_{\mathcal{N}}}^+\}$. In addition, the intrinsic algebra is a deformation of the natural one and vice versa (remember section 2.3). Indeed, by comparing (5.6) with (5.8) one can show that

$$a_{k_{\mathcal{N}}}^- = \frac{r_{\mathcal{N}}(N_k+1)}{r_{\mathcal{I}}(N_k+1)}a_k^-, \qquad a_{k_{\mathcal{N}}}^+ = \frac{r_{\mathcal{N}}(N_k)}{r_{\mathcal{I}}(N_k)}a_k^+, \qquad a_{k_{\mathcal{N}}}^+a_{k_{\mathcal{N}}}^- = [E(N_k) - E_0]\left[\frac{r_{\mathcal{N}}(N_k)}{r_{\mathcal{I}}(N_k)}\right]^2. \tag{5.17}$$

We will see next another deformation of the intrinsic algebra generated by $\{N_k, a_k^-, a_k^+\}$.

6. Linear algebra of H_k

Let us now introduce a new pair of annihilation and creation operators for H_k, such that their action onto the $|\theta_n\rangle$'s is similar to the oscillator one (see (2.15)–(2.16)):

$$a_{k_\mathcal{L}}^-|\theta_n\rangle = r_\mathcal{L}(n)|\theta_{n-1}\rangle, \qquad a_{k_\mathcal{L}}^+|\theta_n\rangle = \bar{r}_\mathcal{L}(n+1)|\theta_{n+1}\rangle,$$
$$a_{k_\mathcal{L}}^\pm|\theta_{\epsilon_i}\rangle = 0, \qquad i = 1,\ldots,q.$$

In the Hubbard representation we have

$$a_{k_\mathcal{L}}^- = r_\mathcal{L}(N_k+1)\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_{m+1}|, \qquad a_{k_\mathcal{L}}^+ = \bar{r}_\mathcal{L}(N_k)\sum_{m=0}^{\infty}|\theta_{m+1}\rangle\langle\theta_m|. \tag{6.1}$$

It is simple to show that

$$\left[N_k, a_{k_\mathcal{L}}^\pm\right] = \pm a_{k_\mathcal{L}}^\pm, \qquad \left[a_{k_\mathcal{L}}^-, a_{k_\mathcal{L}}^+\right] = \sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_m|. \tag{6.2}$$

One can also find that

$$a_{k_\mathcal{L}}^\pm \widetilde{\sum_m}|\theta_m\rangle\langle\theta_m| = C_k^+ a_{0_\mathcal{L}}^\pm C_k. \tag{6.3}$$

By comparing (6.1) with (5.8), it is seen that the linear annihilation and creation operators $a_{k_\mathcal{L}}^\pm$ are deformations of the intrinsic ones $a_k^\pm$ to get a simpler algebra, namely

$$a_{k_\mathcal{L}}^- = \frac{r_\mathcal{L}(N_k+1)}{r_\mathcal{I}(N_k+1)}a_k^-, \qquad a_{k_\mathcal{L}}^+ = \frac{r_\mathcal{L}(N_k)}{r_\mathcal{I}(N_k)}a_k^+, \qquad a_{k_\mathcal{L}}^+ a_{k_\mathcal{L}}^- = N_k. \tag{6.4}$$

7. Coherent states of H_k

Let us construct three sets (in general non-equivalent) of CS as eigenstates of $a_{k_\mathcal{N}}^-, a_k^-, a_{k_\mathcal{L}}^-$. According to the algebra involved, they will be called natural, intrinsic and linear CS, respectively. It will be seen that some differences with respect to the CS of H_0 arise.

7.1. Natural nonlinear coherent states of H_k

We build up first the *natural nonlinear coherent states* $|z,\alpha\rangle_{k_\mathcal{N}}$ which are eigenstates of $a_{k_\mathcal{N}}^-$. Their expansion in terms of eigenstates of H_k reads

$$|z,\alpha\rangle_{k_\mathcal{N}} = \sum_{i=1}^{q}\gamma_{\epsilon_i}|\theta_{\epsilon_i}\rangle + \sum_{m=0}^{\infty}\gamma_m|\theta_m\rangle. \tag{7.1}$$

From the CS definition and making use of (5.3)–(5.4), we get $\gamma_{\epsilon_i} = 0, i = 1,\ldots,q$, and

$$r_\mathcal{N}(m)\gamma_m = z\gamma_{m-1}, \qquad m = 1,2,\ldots. \tag{7.2}$$

According to our SUSY treatment, $\epsilon_s = E_{m_p}$ is the largest eigenvalue of H_k, of the part isospectral to H_0, for which $B_k|\theta_{m_p}\rangle = a_{k_\mathcal{N}}^\pm|\theta_{m_p}\rangle = 0$. Moreover, since $B_k^+|\psi_{m_p}\rangle = 0$ it turns out that $a_{k_\mathcal{N}}^-|\theta_{m_p+1}\rangle = 0$, i.e., $r_\mathcal{N}(m_p+1) = 0$, and by using (7.2) this implies that $\gamma_{m_p} = 0$. By iterating down this equation we arrive at $\gamma_m = 0, m = 0,\ldots,m_p$. Equation (7.2) can be used again to express $\gamma_{m+m_p+1}, m > 0$, in terms of γ_{m_p+1}:

$$\gamma_{m+m_p+1} = \frac{z^m}{r_\mathcal{N}(m+m_p+1)r_\mathcal{N}(m+m_p)\cdots r_\mathcal{N}(m_p+2)}\gamma_{m_p+1}, \qquad m > 0. \tag{7.3}$$

By using the normalization condition and asking for $\gamma_{m_p+1} \in \mathbb{R}^+$, we finally obtain

$$|z,\alpha\rangle_{k_\mathcal{N}} = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\widetilde{\rho}_m}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \mathrm{e}^{-\mathrm{i}\alpha(E_{m+m_p+1}-E_{m_p+1})} \frac{z^m}{\sqrt{\widetilde{\rho}_m}} \left|\theta_{m+m_p+1}\right\rangle, \tag{7.4}$$

where $\widetilde{\rho}_0 = 1$ and, for $m > 0$,

$$\widetilde{\rho}_m = \frac{\rho_{m+m_p+1}}{\rho_{m_p+1}} \prod_{i=1}^{k} \left(E_{m+m_p+1} - \epsilon_i\right)\left(E_{m+m_p} - \epsilon_i\right)^2 \ldots \left(E_{m_p+2} - \epsilon_i\right)^2\left(E_{m_p+1} - \epsilon_i\right), \tag{7.5}$$

with ρ_m given by (3.3).

An important difference of $|z,\alpha\rangle_{k_\mathcal{N}}$ with respect to the sets of CS of H_0 is that the completeness relationship now has to include the eigenstates of H_k which are missing in expansion (7.4), i.e.,

$$\sum_{i=1}^{q} \left|\theta_{\epsilon_i}\right\rangle\!\left\langle\theta_{\epsilon_i}\right| + \sum_{m=0}^{m_p} |\theta_m\rangle\langle\theta_m| + \int |z,\alpha\rangle_{k_\mathcal{N}\,k_\mathcal{N}}\langle z,\alpha|\,\mathrm{d}\widetilde{\mu}(z) = 1. \tag{7.6}$$

A similar procedure as for the CS of H_0 leads to

$$\mathrm{d}\widetilde{\mu}(z) = \frac{1}{\pi}\left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\widetilde{\rho}_m}\right)\widetilde{\rho}(|z|^2)\,\mathrm{d}^2 z, \tag{7.7}$$

$\widetilde{\rho}(y)$ satisfying a moment problem more complicated than the initial one (compare ρ_m of (3.3) with $\widetilde{\rho}_m$ of (7.5)):

$$\int_0^\infty y^m \widetilde{\rho}(y)\,\mathrm{d}y = \widetilde{\rho}_m, \qquad m \geqslant 0. \tag{7.8}$$

Another relevant difference is that, since $B_k\left|\theta_{\epsilon_i}\right\rangle = a_{k_\mathcal{N}}^-\left|\theta_{\epsilon_i}\right\rangle = 0, i = 1,\ldots,q$, $B_k\left|\theta_{m_j}\right\rangle = a_{k_\mathcal{N}}^-\left|\theta_{m_j}\right\rangle = 0$, $a_{k_\mathcal{N}}^-\left|\theta_{m_j+1}\right\rangle = 0$, $j = 1,\ldots p$, and $a_{k_\mathcal{N}}^-|\theta_0\rangle = 0$, then the degeneracy of the eigenvalue $z = 0$ of $a_{k_\mathcal{N}}^-$ can be any integer in the set $\{s+1,\ldots,s+p+1\}$, depending on the positions of the levels E_{m_j}, $j = 1,\ldots,p$. However, once again by the phase choice of equation (2.6), the natural CS $|z,\alpha\rangle_{k_\mathcal{N}}$ of (7.4) evolve coherently in time:

$$U_k(t)|z,\alpha\rangle_{k_\mathcal{N}} = \mathrm{e}^{-\mathrm{i}tE_{m_p+1}}|z,\alpha+t\rangle_{k_\mathcal{N}}, \tag{7.9}$$

$U_k(t) = \exp(-\mathrm{i}tH_k)$ being the evolution operator associated with H_k. This property also will be valid for the other CS of H_k which will be further derived.

Let us remark that some properties of the natural nonlinear CS of H_k were studied previously for the SUSY partners of the harmonic oscillator [29, 34]. To compare with the case discussed in [34], let us restrict ourselves to SUSY transformations for which the seeds are just nonphysical eigenfunctions of H_0, i.e., take $p = 0$ and $q = s \leqslant k$. Now the only eigenstate of H_k for the part of the spectrum isospectral to H_0 which is annihilated by $a_{k_\mathcal{N}}^-$ is $|\theta_0\rangle$, and thus the CS expansion (7.4) should start from this state. This is achieved by defining $m_{p=0} = -1$: with this choice and taking the harmonic oscillator energy levels in the CS of (7.4) one arrives at the CS of equation (5.14) in [34].

7.2. Intrinsic nonlinear coherent states of H_k

Let us analyse next the intrinsic nonlinear CS $|z,\alpha\rangle_k$ which are eigenstates of a_k^-. A similar procedure as before leads to

$$|z,\alpha\rangle_k = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \mathrm{e}^{-\mathrm{i}\alpha(E_m-E_0)} \frac{z^m}{\sqrt{\rho_m}}|\theta_m\rangle. \tag{7.10}$$

This expansion is also obtained from the intrinsic nonlinear CS $|z, \alpha\rangle_0$ of H_0 and vice versa by the change $|\psi_n\rangle \leftrightarrow |\theta_n\rangle$ (cf equations (3.2) and (7.10)). Thus, the completeness relationship is automatically satisfied,

$$\sum_{i=1}^{q} |\theta_{\epsilon_i}\rangle\langle\theta_{\epsilon_i}| + \int |z, \alpha\rangle_{k\,k}\langle z, \alpha|\, \mathrm{d}\mu(z) = 1, \tag{7.11}$$

where $\mathrm{d}\mu(z)$ is given by equations (3.5), (3.6). This is a simplification with respect to the natural nonlinear CS $|z, \alpha\rangle_{k_\mathcal{N}}$ of (7.4), (7.5). After some simple manipulations we also arrive at

$$|z, \alpha\rangle_k = C_k^+ |z, \alpha\rangle_0 + \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}} \sum_{j=1}^{p} \mathrm{e}^{-\mathrm{i}\alpha(E_{m_j}-E_0)} \frac{z^{m_j}}{\sqrt{\rho_{m_j}}} |\theta_{m_j}\rangle. \tag{7.12}$$

Since $a_k^- |\theta_{\epsilon_i}\rangle = 0,\ i = 1, \ldots, q$ and taking into account that

$$|z = 0, \alpha\rangle_k = |\theta_0\rangle, \tag{7.13}$$

it turns out that the eigenvalue $z = 0$ of a_k^- is $(q + 1)$th degenerated.

7.3. Linear coherent states of H_k

Let us consider the linear CS which are eigenstates of $a_{k_\mathcal{L}}^-$. Since the algebra of $a_{k_\mathcal{L}}^\pm$ acting onto $\mathrm{Span}\{|\theta_n\rangle, n = 0, 1, \ldots\}$ is equal to that of $a_{0_\mathcal{L}}^\pm$ acting onto $\mathrm{Span}\{|\psi_n\rangle, n = 0, 1, \ldots\}$, it can be shown that

$$|z, \alpha\rangle_{k_\mathcal{L}} = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \mathrm{e}^{-\mathrm{i}\alpha(E_m-E_0)} \frac{z^m}{\sqrt{m!}} |\theta_m\rangle. \tag{7.14}$$

This expression is also obtained from the corresponding one for $|z, \alpha\rangle_{0_\mathcal{L}}$ and vice versa by the mapping $|\psi_m\rangle \leftrightarrow |\theta_m\rangle$ (cf (3.12) and (7.14)). Thus, the completeness relationship is identified in a simple way:

$$\sum_{i=1}^{q} |\theta_{\epsilon_i}\rangle\langle\theta_{\epsilon_i}| + \frac{1}{\pi}\int |z, \alpha\rangle_{k_\mathcal{L}\,k_\mathcal{L}}\langle z, \alpha|\, \mathrm{d}^2 z = 1. \tag{7.15}$$

Moreover, it turns out that

$$|z, \alpha\rangle_{k_\mathcal{L}} = C_k^+ |z, \alpha\rangle_{0_\mathcal{L}} + \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{j=1}^{p} \mathrm{e}^{-\mathrm{i}\alpha(E_{m_j}-E_0)} \frac{z^{m_j}}{\sqrt{m_j!}} |\theta_{m_j}\rangle. \tag{7.16}$$

The eigenvalue $z = 0$ of $a_{k_\mathcal{L}}^-$ is $(q + 1)$th degenerated, a property discovered for the first time for the SUSY partners of the harmonic oscillator [29, 34]. It can also be found that

$$|z, \alpha\rangle_{k_\mathcal{L}} = D_{k_\mathcal{L}} |\theta_0\rangle = \exp\left(z a_{k_\mathcal{L}}^+ - \bar{z} a_{k_\mathcal{L}}^-\right) |\theta_0\rangle. \tag{7.17}$$

8. Examples

We will apply the previous techniques to some examples: the harmonic oscillator, infinite square well and trigonometric Pöschl–Teller potentials. For each system we will use a different kind of SUSY transformation, depending on how many physical eigenstates $|\theta_{\epsilon_i}\rangle$ of H_k which are annihilated by B_k have energies different from those of H_0. Thus, for the harmonic oscillator we will study the general situation with $q \neq 0$, $p \neq 0$, while for the infinite square well the strictly isospectral case with $q = 0$, $p = s$ will be explored. For the Pöschl–Teller potential the s levels ϵ_i will be different from those of H_0 (i.e. for $q = s$, $p = 0$).

8.1. The harmonic oscillator

Let us consider the harmonic oscillator potential:

$$V_0(x) = \frac{x^2}{2}. \tag{8.1}$$

The normalized eigenfunctions and eigenvalues of H_0 are given by

$$\psi_n(x) = \langle x|\psi_n\rangle = \frac{\mathrm{e}^{-\frac{x^2}{2}} H_n(x)}{\sqrt{\sqrt{\pi}2^n n!}}, \qquad E(n) \equiv E_n = n + \frac{1}{2}, \qquad n = 0, 1, \ldots \tag{8.2}$$

where $H_n(x)$ are the Hermite polynomials. Since $E(n)$ is linear in n, it is simple to show that $f(N_0) = 1$. Thus, after dropping some unimportant global phases, the *intrinsic* algebra reduces to the Heisenberg–Weyl one, as was expected. This implies that the corresponding CS as well become the canonical ones (take $\alpha = 0$ in the formulae of sections 2.1 and 3.2).

8.1.1. The SUSY partners H_k. Let us study the kth order SUSY partners of the harmonic oscillator. In order to implement the transformation, we look for the general solution $u(x)$ of the stationary Schrödinger equation (4.6) with the oscillator potential (8.1) for an arbitrary factorization energy ϵ. Up to a constant factor we obtain

$$u(x) = \mathrm{e}^{-\frac{x^2}{2}} \left[{}_1F_1\left(\frac{1}{4} - \frac{\epsilon}{2}; \frac{1}{2}; x^2\right) + 2\mu x \frac{\Gamma\left(\frac{3}{4} - \frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{\epsilon}{2}\right)} {}_1F_1\left(\frac{3}{4} - \frac{\epsilon}{2}; \frac{3}{2}; x^2\right)\right], \tag{8.3}$$

where ${}_1F_1(a; b; y)$ is the confluent hypergeometric function and $u(x)$ is nodeless for $\epsilon < 1/2, |\mu| < 1$ [34]. By using this expression to specify the seed solutions, the associated Wronskian can be calculated, which automatically leads to the new potential and the corresponding energy eigenstates.

8.1.2. Algebraic structures of H_k. The annihilation and creation operators for the several algebras of H_k, in terms of the intrinsic ones $a_k^{\pm}$, are given by equations (5.17), (6.4), where

$$\frac{r_{\mathcal{N}}(n)}{r_{\mathcal{I}}(n)} = \left[\prod_{i=1}^{k} \left(n - \epsilon_i - \frac{1}{2}\right)\left(n - \epsilon_i + \frac{1}{2}\right)\right]^{\frac{1}{2}}, \qquad \frac{r_{\mathcal{L}}(n)}{r_{\mathcal{I}}(n)} = 1. \tag{8.4}$$

Up to a global phase factor, the intrinsic operators $a_k^{\pm}$ are those of (5.8) with $r_{\mathcal{I}}(n) = \sqrt{n}$, i.e., we recover the Heisenberg–Weyl algebra onto Span$\{|\theta_n\rangle, n = 0, 1, \ldots\}$.

8.1.3. Coherent states of H_k. In order to find the natural nonlinear CS of H_k, we determine first the coefficients $\widetilde{\rho}_m$ of (7.4), (7.5):

$$\widetilde{\rho}_m = (m_p + 2)_m \prod_{i=1}^{k} \left(m_p - \epsilon_i + \frac{3}{2}\right)_m \left(m_p - \epsilon_i + \frac{5}{2}\right)_m, \qquad m \geqslant 0, \tag{8.5}$$

with the Pochhammer symbol given by $(b)_m = \Gamma(b + m)/\Gamma(b)$. Hence we get

$$|z, \alpha\rangle_{k_{\mathcal{N}}} = \frac{1}{\sqrt{{}_1F_{2k+1}\left(1; m_p + 2, \ldots, m_p - \epsilon_i + \frac{3}{2}, m_p - \epsilon_i + \frac{5}{2}, \ldots; |z|^2\right)}} \times \sum_{m=0}^{\infty} \frac{z^m |\theta_{m+m_p+1}\rangle}{\sqrt{(m_p + 2)_m} \prod_{i=1}^{k} \sqrt{\left(m_p - \epsilon_i + \frac{3}{2}\right)_m \left(m_p - \epsilon_i + \frac{5}{2}\right)_m}}, \tag{8.6}$$

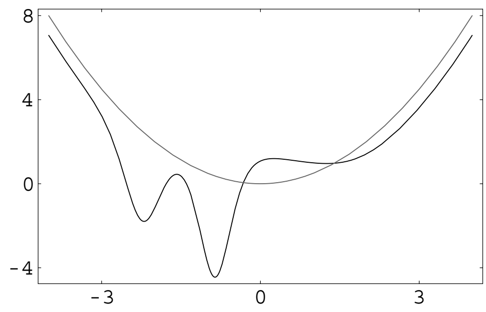

Figure 1. Third-order SUSY partner potential $V_3(x)$ (black curve) of the oscillator (grey curve) obtained by composing a confluent second-order transformation with seed the ground state of H_0 ($w_0 = 0.51$) and a first-order one with $\epsilon_1 = -3/2$ ($\mu = 0.99$). The net result is the 'creation' of an energy level at ϵ_1 for H_3.

where ${}_pF_q$ is a generalized hypergeometric function defined by

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{m=0}^{\infty} \frac{(a_1)_m \ldots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{x^m}{m!}. \tag{8.7} $$

It is clear that the moment problem (7.8) with the $\widetilde{\rho}_m$ of (8.5) is more involved than the already solved initial one, and it can be worked out once the factorization energies ϵ_i are specified. Indeed, a few solutions for some SUSY transformations have been derived elsewhere [29, 34].

For the intrinsic nonlinear and linear CS of H_k, both expressions are the same and coincide with the canonical expansion, which arises from (3.12) for $\alpha = 0$ with the change $|\psi_m\rangle \to |\theta_m\rangle$.

In particular, we illustrate the SUSY partner potential $\widetilde{V}_3(x)$ generated from a third-order transformation with $k = 3, q = p = 1$. The seeds u_1, u_2, u_3 correspond to solution (8.3) with $\epsilon_1 = -3/2$ for u_1, the ground state eigenfunction $\psi_0(x)$ of (8.2) with $\epsilon_2 = E_0 = 1/2$ for u_2, and a generalized eigenfunction of second order associated with $\epsilon_3 = \epsilon_2$ for u_3 such that $(H_0 - \epsilon_2)u_3 = u_2 \Rightarrow (H_0 - \epsilon_2)^2 u_3 = 0$, its nontrivial part given by [46]

$$ u_3 = \frac{e^{-\frac{x^2}{2}}}{2\pi^{\frac{1}{4}}} \left[\pi w_0 \, \mathrm{Erfi}(x) + x^2 {}_2F_2\left(1, 1; \frac{3}{2}, 2; x^2\right) \right]. \tag{8.8} $$

The new potential is obtained from (4.7), with the Wronskian expressed as

$$ \begin{aligned} W(u_1, u_2, u_3) = \frac{e^{-\frac{3x^2}{2}}}{\sqrt{\pi}} \big\{ & -2x + 4\pi w_0 \mu x \, e^{2x^2} + \sqrt{\pi} \, e^{x^2} \big[4w_0 - \mu - 2\mu x^2 \\ & + \big(1 + 2\sqrt{\pi}(\mu + 2w_0) x \, e^{x^2} - 2x^2\big) \mathrm{Erf}(x) \big] + 2\pi x \, e^{2x^2} [\mathrm{Erf}(x)]^2 \big\}. \end{aligned} \tag{8.9} $$

This Wronskian is nodeless for $|\mu| < 1$ and $|w_0| > 1/2$. A member of the family of potentials (4.7) is shown in figure 1 for $\mu = 0.99$ and $w_0 = 0.51$. The spectrum of the Hamiltonian H_3 is $\{\epsilon_1 = -3/2, E_n = n + 1/2, n = 0, 1, \ldots\}$.

8.2. The infinite well potential

In dimensionless units, the infinite well potential we shall study reads

$$ V_0(x) = \begin{cases} \infty & \text{for} \quad x = 0, \pi, \\ 0 & \text{for} \quad 0 < x < \pi. \end{cases} \tag{8.10} $$

The eigenfunctions and eigenvalues are well known:

$$\psi_n(x) = \sqrt{\frac{2}{\pi}}\sin[(n+1)x], \qquad E_n = E(n) = \frac{(n+1)^2}{2}, \qquad n = 0, 1, \ldots. \tag{8.11}$$

8.2.1. Intrinsic algebra of H_0. It is determined by the operator function

$$E(N_0) = \frac{(N_0+1)^2}{2} = H_0, \tag{8.12}$$

leading thus to the following structure function:

$$f(N_0) = E(N_0+1) - E(N_0) = N_0 + \tfrac{3}{2}. \tag{8.13}$$

The Hubbard representation for the intrinsic operators $a_0^\pm$ is given by (2.13), where now

$$r_{\mathcal{I}}(n) = \mathrm{e}^{\mathrm{i}\alpha(n+\frac{1}{2})}\sqrt{\frac{n(n+2)}{2}}. \tag{8.14}$$

The operator set $\{N_0, a_0^-, a_0^+\}$ then satisfies the commutation relationships

$$[N_0, a_0^\pm] = \pm a_0^\pm, \qquad [a_0^-, a_0^+] = N_0 + \tfrac{3}{2}, \tag{8.15}$$

which, after redefining the number operator as $\widetilde{N}_0 = N_0 + \frac{3}{2}$, reduce to the $su(1,1)$ algebra.

8.2.2. Linear algebra of H_0. The linear operators $a_{0_\mathcal{L}}^\pm$, expressed as deformations of the intrinsic ones $a_0^\pm$, acquire the form

$$a_{0_\mathcal{L}}^- = \sqrt{\frac{2}{N_0+3}}\,a_0^-, \qquad a_{0_\mathcal{L}}^+ = a_0^+\sqrt{\frac{2}{N_0+3}}, \qquad a_{0_\mathcal{L}}^+ a_{0_\mathcal{L}}^- = N_0. \tag{8.16}$$

By construction, their action onto the eigenstates of H_0 is the standard one (up to some phase factors).

8.2.3. Coherent states of H_0. The intrinsic nonlinear and linear CS of H_0 become

$$|z,\alpha\rangle_0 = [{}_0F_1(3; 2|z|^2)]^{-\frac{1}{2}}\sum_{m=0}^{\infty}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}m(m+2)}\sqrt{\frac{2^{m+1}}{m!(m+2)!}}\,z^m|\psi_m\rangle, \tag{8.17}$$

$$|z,\alpha\rangle_{0_\mathcal{L}} = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{m=0}^{\infty}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}m(m+2)}\frac{z^m}{\sqrt{m!}}|\psi_m\rangle. \tag{8.18}$$

The completeness of the intrinsic nonlinear CS (8.17) is ensured since the moment problem (3.6) with $\rho_m = m!(m+2)!/2^{m+1}$ admits the positive definite solution

$$\rho(y) = 4yK_2(2\sqrt{2y}), \tag{8.19}$$

with $K_2(y)$ being a modified Bessel function of second kind. Hence, the measure (3.5) reads

$$\mathrm{d}\mu(z) = \frac{4|z|^2}{\pi}\,{}_0F_1(3; 2|z|^2)K_2(2\sqrt{2}|z|)\,\mathrm{d}^2z. \tag{8.20}$$

The reproducing kernel (3.8) acquires the form

$${}_0\langle z,\alpha|z',\alpha\rangle_0 = [{}_0F_1(3; 2|z|^2)\,{}_0F_1(3; 2|z'|^2)]^{-\frac{1}{2}}\,{}_0F_1(3; 2\bar{z}z'). \tag{8.21}$$

On the other hand, for the linear CS (8.18) directly apply the formulae of section 3.2, in particular the completeness relationship (3.13) and the reproducing kernel (3.15).

8.2.4. The SUSY partners H_k. For generating the kth order SUSY partners of the infinite well potential, we employ isospectral transformations which do not create new levels. This implies that $q = 0$, $p = s \leqslant k$, and there are p levels of H_0, $\epsilon_j = E_{m_j} = (m_j + 1)^2/2$, $j = 1, \ldots, p$, whose physical eigenstates $|\psi_{m_j}\rangle$ are annihilated by B_k^+ and will be used as seeds to implement the procedure.

8.2.5. Algebraic structures of H_k. The natural and linear annihilation and creation operators of H_k, in terms of the intrinsic ones $a_k^{\pm}$, are written in equations (5.17), (6.4), where

$$\frac{r_{\mathcal{N}}(n)}{r_{\mathcal{I}}(n)} = 2^{-k}\prod_{i=1}^{k}\sqrt{[n^2 - 2\epsilon_i][(n+1)^2 - 2\epsilon_i]}, \qquad \frac{r_{\mathcal{L}}(n)}{r_{\mathcal{I}}(n)} = \sqrt{\frac{2}{n+2}}. \tag{8.22}$$

The intrinsic operators are given in equation (5.8) with $r_{\mathcal{I}}(n)$ given by (8.14).

8.2.6. Coherent states of H_k. The coefficients $\widetilde{\rho}_m$ in (7.4), (7.5), required to find the natural nonlinear CS $|z, \alpha\rangle_{k_{\mathcal{N}}}$, take the form

$$\widetilde{\rho}_m = \frac{(m_p+2)_m(m_p+4)_m}{2^{m(2k+1)}}\prod_{i=1}^{k}(m_p - \sqrt{2\epsilon_i} + 2)_m(m_p - \sqrt{2\epsilon_i} + 3)_m \times (m_p + \sqrt{2\epsilon_i} + 2)_m(m_p + \sqrt{2\epsilon_i} + 3)_m, \qquad m \geqslant 0. \tag{8.23}$$

Therefore

$$|z,\alpha\rangle_{k_{\mathcal{N}}} = \frac{1}{\sqrt{{}_1F_{4k+2}(1; m_p+2, m_p+4, \ldots, m_p-\sqrt{2\epsilon_i}+2, m_p-\sqrt{2\epsilon_i}+3, m_p+\sqrt{2\epsilon_i}+2, m_p+\sqrt{2\epsilon_i}+3, \ldots; 2^{2k+1}|z|^2)}} \times \sum_{m=0}^{\infty}\frac{e^{-\frac{i}{2}\alpha m(m+2m_p+4)}\sqrt{2^{m(2k+1)}}\,z^m\,|\theta_{m+m_p+1}\rangle}{\sqrt{(m_p+2)_m(m_p+4)_m}\prod_{i=1}^{k}\sqrt{(m_p-\sqrt{2\epsilon_i}+2)_m(m_p-\sqrt{2\epsilon_i}+3)_m(m_p+\sqrt{2\epsilon_i}+2)_m(m_p+\sqrt{2\epsilon_i}+3)_m}}. \tag{8.24}$$

The moment problem (7.8) with the $\widetilde{\rho}_m$ of (8.23) can be worked out once the factorization energies $\epsilon_1, \ldots, \epsilon_k$ are specified. These quantities determine as well the degeneracy of the eigenvalue $z = 0$ of $a_{k_{\mathcal{N}}}$, which can take a value in the set $\{p+1, \ldots, 2p+1\}$.

The intrinsic nonlinear and linear CS of H_k are obtained from (8.17) and (8.18) respectively by the replacement $|\psi_m\rangle \to |\theta_m\rangle$.

For illustrating some isospectral SUSY partners of the infinite well (8.10), we employ a confluent second-order transformation involving one physical eigenfunction of H_0, i.e., we take $k = 2$, $\epsilon_1 = \epsilon_2 = E_{m_1} = (m_1 + 1)^2/2$ [45, 46]. We need to evaluate the Wronskian of two generalized eigenfunctions u_1, u_2 of H_0: u_1 is the standard physical eigenfunction $\psi_{m_1}(x)$ of (8.11) obeying $(H_0 - \epsilon_1)u_1 = 0$, but u_2 is a second-order generalized eigenfunction such that $(H_0 - \epsilon_1)u_2 = u_1 \Rightarrow (H_0 - \epsilon_1)^2 u_2 = 0$ [46]. The expression for u_2 is

$$u_2(x) = -\frac{(\pi w_0 + x)}{\sqrt{2\pi}(m_1+1)}\cos[(m_1+1)x]. \tag{8.25}$$

This allows us to evaluate the Wronskian $W(u_1, u_2)$, and then the new potential,

$$V_2(x) = \begin{cases}\infty & \text{for} \quad x = 0, \pi, \\ \frac{16(m_1+1)^2\sin[(m_1+1)x]\{\sin[(m_1+1)x]-(m_1+1)(\pi w_0+x)\cos[(m_1+1)x]\}}{\{\sin[2(m_1+1)x]-2(m_1+1)(\pi w_0+x)\}^2} & \text{for} \quad 0 < x < \pi,\end{cases} \tag{8.26}$$

which is non-singular for $x \in (0, \pi)$ if $w_0 > 0$ or $w_0 < -1$. An example of these potentials is shown in figure 2 for $m_1 = 1$, $w_0 = 0.1$ (black curve), where the infinite well (8.10) is drawn in grey.

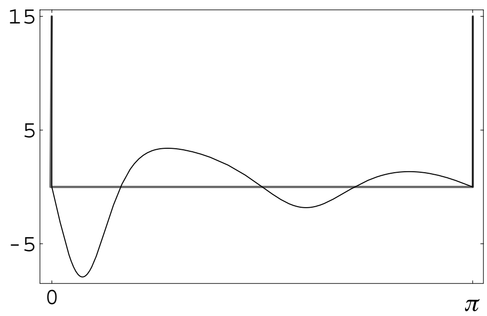

Figure 2. Second-order SUSY partner potential $V_2(x)$ (black curve) isospectral to the infinite well (grey line) obtained by a confluent second-order transformation involving the eigenfunction of the first excited state of H_0 and $w_0 = 0.1$.

8.3. The trigonometric Pöschl–Teller potential

In appropriate units the trigonometric Pöschl–Teller potential can be written as

$$V_0(x) = \frac{\nu(\nu-1)}{2\cos^2(x)}, \qquad \nu > 1. \tag{8.27}$$

The energy eigenstates $\psi_n(x)$ are expressed in terms of Gegenbauer polynomials $C_n^\nu(y)$ while the eigenvalues are quadratic in n [12, 48]:

$$\begin{aligned} \psi_n(x) &= \left[\frac{n!(n+\nu)\Gamma(\nu)\Gamma(2\nu)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(n+2\nu)}\right]^{1/2} \cos^\nu(x) C_n^\nu(\sin(x)), \\ E_n &= E(n) = \frac{(n+\nu)^2}{2}, \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{8.28}$$

8.3.1. Intrinsic algebra of H_0. It is defined by

$$E(N_0) = \frac{(N_0+\nu)^2}{2} = H_0, \tag{8.29}$$

giving place to the following structure function:

$$f(N_0) = E(N_0+1) - E(N_0) = N_0 + \nu + \tfrac{1}{2}. \tag{8.30}$$

The Hubbard representation for the intrinsic operators $a_0^\pm$ is given again by (2.13) with

$$r_{\mathcal{I}}(n) = \mathrm{e}^{\mathrm{i}\alpha(n+\nu-\frac{1}{2})}\sqrt{\frac{n(n+2\nu)}{2}}. \tag{8.31}$$

The operator set $\{N_0, a_0^-, a_0^+\}$ satisfies the commutation relationships,

$$[N_0, a_0^\pm] = \pm a_0^\pm, \qquad [a_0^-, a_0^+] = N_0 + \nu + \tfrac{1}{2}, \tag{8.32}$$

which, redefining the number operator as $\widetilde{N}_0 = N_0 + \nu + \frac{1}{2}$, reduce to the $su(1,1)$ algebra.

8.3.2. Linear algebra of H_0. The linear annihilation and creation operators $a_{0_\mathcal{L}}^\pm$ can be expressed as deformations of the intrinsic ones $a_0^\pm$:

$$a_{0_\mathcal{L}}^- = \sqrt{\frac{2}{N_0+2\nu+1}}\, a_0^-, \qquad a_{0_\mathcal{L}}^+ = a_0^+\sqrt{\frac{2}{N_0+2\nu+1}}, \qquad a_{0_\mathcal{L}}^+ a_{0_\mathcal{L}}^- = N_0. \tag{8.33}$$

Once again, by construction they act on the eigenstates of H_0 in a standard way (up to some phase factors).

8.3.3. Coherent states of H_0. The intrinsic nonlinear and linear CS now become

$$|z,\alpha\rangle_0 = [{}_0F_1(2\nu+1;2|z|^2)]^{-\frac{1}{2}}\sum_{m=0}^{\infty}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}m(m+2\nu)}\sqrt{\frac{2^m}{m!(2\nu+1)_m}}z^m|\psi_m\rangle, \tag{8.34}$$

$$|z,\alpha\rangle_{0_{\mathcal{L}}} = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{m=0}^{\infty}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}m(m+2\nu)}\frac{z^m}{\sqrt{m!}}|\psi_m\rangle. \tag{8.35}$$

The set of intrinsic nonlinear CS (8.34) is complete since the moment problem (3.6) with

$$\rho_m = \frac{m!(2\nu+1)_m}{2^m} \tag{8.36}$$

can be simply solved, with a positive definite function $\rho(y)$ given by

$$\rho(y) = \frac{2^{\nu+2}y^{\nu}}{\Gamma(2\nu+1)}K_{2\nu}(2\sqrt{2y}). \tag{8.37}$$

Hence, the invariant measure (3.5) becomes

$$\mathrm{d}\mu(z) = \frac{2^{\nu+2}|z|^{2\nu}}{\pi\Gamma(2\nu+1)}{}_0F_1(2\nu+1;2|z|^2)K_{2\nu}(2\sqrt{2}|z|)\,\mathrm{d}^2z. \tag{8.38}$$

The reproducing kernel (3.8) reads

$${}_0\langle z,\alpha|z',\alpha\rangle_0 = [{}_0F_1(2\nu+1;2|z|^2)\,{}_0F_1(2\nu+1;2|z'|^2)]^{-\frac{1}{2}}\,{}_0F_1(2\nu+1;2\bar{z}z'). \tag{8.39}$$

For the linear CS (8.35) of H_0 all formulae of section 3.2 become the same, so we skipped them, as we did for the infinite well potential (8.10).

8.3.4. The SUSY partners H_k. For generating the kth order SUSY partners of the Pöschl–Teller potential (8.27), we use transformations involving just seed solutions associated with non-physical factorization energies $\epsilon_i, i = 1,\ldots,k$, of H_0, q of them becoming physical levels of H_k. The general mathematical eigenfunction $u(x)$ of H_0 for arbitrary ϵ is given by

$$u(x) = \cos^{\nu}(x)\left[{}_2F_1\left(\frac{\nu}{2}-\sqrt{\frac{\epsilon}{2}},\frac{\nu}{2}+\sqrt{\frac{\epsilon}{2}};\frac{1}{2};\sin^2(x)\right)\right.$$
$$\left.+\,\mu\sin(x)\,{}_2F_1\left(\frac{\nu}{2}+\sqrt{\frac{\epsilon}{2}}+\frac{1}{2},\frac{\nu}{2}-\sqrt{\frac{\epsilon}{2}}+\frac{1}{2};\frac{3}{2};\sin^2(x)\right)\right]. \tag{8.40}$$

This expression supplies any seed solution involved in the Wronskian of the transformation, which leads to the potential $V_k(x)$ as well as the eigenstates of H_k.

8.3.5. Algebraic structures of H_k. The annihilation and creation operators for the natural and linear algebras of H_k are written in equations (5.17), (6.4) with

$$\frac{r_{\mathcal{N}}(n)}{r_{\mathcal{I}}(n)} = 2^{-k}\prod_{i=1}^{k}\sqrt{[(n+\nu-1)^2-2\epsilon_i][(n+\nu)^2-2\epsilon_i]}, \qquad \frac{r_{\mathcal{L}}(n)}{r_{\mathcal{I}}(n)} = \sqrt{\frac{2}{n+2\nu}}. \tag{8.41}$$

The intrinsic operators are given in equation (5.8) with $r_{\mathcal{I}}(n)$ given by (8.31).

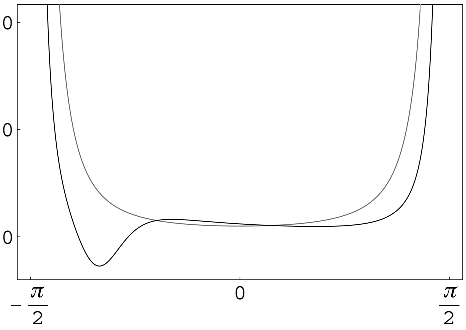

Figure 3. First-order SUSY partner potential $V_1(x)$ (black curve) of the Pöschl–Teller potential with $\nu = 3$ (grey curve) obtained by using as seed the $u(x)$ of (8.40) with $\mu = 1.9, \epsilon = 3/2 < E_0 = 9/2$. The new potential has an additional level at ϵ.

8.3.6. Coherent states of H_k. The coefficients $\widetilde{\rho}_m$ of (7.4), (7.5) required to find the natural nonlinear CS of H_k are now

$$\widetilde{\rho}_m = \frac{m!(2\nu+1)_m}{2^{m(2k+1)}} \prod_{i=1}^{k} (\nu - \sqrt{2\epsilon_i})_m (\nu - \sqrt{2\epsilon_i} + 1)_m (\nu + \sqrt{2\epsilon_i})_m (\nu + \sqrt{2\epsilon_i} + 1)_m, \tag{8.42}$$

where $m \geqslant 0$. Therefore

$$|z, \alpha\rangle_{k_\mathcal{N}} = \frac{1}{\sqrt{{}_0F_{4k+1}(2\nu+1, \ldots, \nu-\sqrt{2\epsilon_i}, \nu-\sqrt{2\epsilon_i}+1, \nu+\sqrt{2\epsilon_i}, \nu+\sqrt{2\epsilon_i}+1, \ldots; 2^{2k+1}|z|^2)}} \times \sum_{m=0}^{\infty} \frac{e^{-\frac{i}{2}\alpha m(m+2\nu)} \sqrt{2^{m(2k+1)}} z^m |\theta_m\rangle}{\sqrt{m!(2\nu+1)_m} \prod_{i=1}^{k} \sqrt{(\nu-\sqrt{2\epsilon_i})_m (\nu-\sqrt{2\epsilon_i}+1)_m (\nu+\sqrt{2\epsilon_i})_m (\nu+\sqrt{2\epsilon_i}+1)_m}}. \tag{8.43}$$

The moment problem (7.8) with the $\widetilde{\rho}_m$ of (8.42) can be worked out once $\epsilon_1, \ldots, \epsilon_k$ are specified. However, the degeneracy of the eigenvalue $z = 0$ of $a_{k_\mathcal{N}}$ is $q + 1$.

The intrinsic nonlinear and linear CS of H_k are obtained from the corresponding ones of H_0 (see (8.34)–(8.35)) by the replacement $|\psi_m\rangle \to |\theta_m\rangle$.

As an illustration, a first-order SUSY transformation which 'creates' a new level at ϵ for H_1 is taken (for $k = q = 1$, $p = 0$). The 'Wronskian' is directly the solution $u(x)$ of (8.40); with this input for $\mu = 1.9$, $\epsilon = 3/2 < E_0 = 9/2$ we have drawn in figure 3 the SUSY partner potential (black curve) of the Pöschl–Teller potential with $\nu = 3$ (grey curve).

9. Conclusions

In this paper we have derived coherent states for Hamiltonians H_k attained from a given initial one through the higher-order SUSY QM. We have shown here, and previously for the harmonic oscillator [29, 34], that it is important to determine the algebraic structures ruling those potentials. It turns out that the intrinsic and linear algebras of the initial Hamiltonian are inherited by its corresponding SUSY partners in the subspace associated with the isospectral part of the spectrum. Moreover, we have discussed an interesting additional algebra of H_k (the so-called natural) generalizing the one which was first introduced for the SUSY partners of the harmonic oscillator [29, 34]. We have shown as well that the natural and intrinsic algebras are deformations from each other, and our analysis shows that the natural is more involved that the intrinsic one. On the other hand, the linear algebra we have studied is a deformation simplifying at maximum the intrinsic structure of our systems. It is worthwhile to note that, up to this moment, the last procedure has been elaborated at a purely algebraic level, and it has been implemented to somehow map the original system into the harmonic oscillator. This suggests a class of problems which could be addressed in the future, in particular, it would be important to analyse the consequences of this linearization at a differential level. This is a quite interesting problem which, as far as we know, is open.

Acknowledgments

The authors acknowledge the support of Conacyt, projects 49253-F and 50766, and research grants from NSERC of Canada. Part of this work has been done while VH visited the Cinvestav and DJFC the Université de Montréal. These institutions are acknowledged for hospitality and financial support.

References

[1] Glauber R J 2006 *Chem. Phys. Chem.* **7** 1618
[2] Klauder J R and Skagerstam B S (ed) 1985 *Coherent States. Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[3] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Heidelberg: Springer)
[4] Zhang W M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
[5] Ali S T, Antoine J P and Gazeau J P 2000 *Coherent States, Wavelets and Their Generalizations* (New York: Springer)
[6] Dodonov V V 2002 *J. Opt.* B **4** R1
Dodonov V V and Manko V I (ed) 2003 *Theory of Non-Classical States of Light* (New York: Taylor and Francis)
[7] Nieto L M 2006 *AIP Conf. Proc.* **809** 3
[8] Spiridonov V 1995 *Phys. Rev.* A **52** 1909
[9] deMatos R L and Vogel W 1996 *Phys. Rev.* A **54** 4560
[10] Manko V I, Marmo G, Sudarshan E C G and Zaccaria F 1997 *Phys. Scr.* **55** 528
Manko O V 1997 *Phys. Lett.* A **228** 29
[11] Roy B and Roy P 1999 *J. Opt.* B **1** 341
Roy B and Roy P 2000 *J. Opt.* B **2** 65
[12] Quesne C 1999 *J. Phys. A: Math. Gen.* **32** 6705
Quesne C 2001 *Ann. Phys.* **293** 147
[13] Sivakumar S 2000 *J. Opt.* B **2** R61
[14] Sunilkumar V, Bambah B A, Jagannathan R, Panigrahi P K and Srinivasan V 2000 *J. Opt.* B **2** 126
[15] Roknizadeh R and Tavassoly M K 2004 *J. Phys. A: Math. Gen.* **37** 8111
[16] Hussin V and Nieto L M 2005 *J. Math. Phys.* **46** 122102
[17] Andrianov A A, Ioffe M V and Spiridonov V 1993 *Phys. Lett.* A **174** 273
Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1995 *Int. J. Mod. Phys.* A **10** 2683
[18] Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 374
Samsonov B F 1999 *Phys. Lett.* A **263** 274
[19] Fernández D J 1997 *Int. J. Mod. Phys.* A **12** 171
Fernández D J, Hussin V and Mielnik B 1998 *Phys. Lett.* A **244** 309
Cariñena J F, Ramos A and Fernández D J 2001 *Ann. Phys.* **292** 42
[20] Bagchi B, Ganguly A, Bhaumik D and Mitra A 1999 *Mod. Phys. Lett.* A **14** 27
Bagchi B K 2001 *Supersymmetry in Quantum and Classical Mechanics* (Boca Raton, FL: Chapman and Hall/CRC Press)
[21] Aoyama H, Sato M and Tanaka T 2001 *Phys. Lett.* B **503** 423
Aoyama H, Sato M and Tanaka T 2001 *Nucl. Phys.* B **619** 105
[22] Andrianov A A and Sokolov A V 2003 *Nucl. Phys.* B **660** 25
Andrianov A A and Cannata F 2004 *J. Phys. A: Math. Gen.* **37** 10297
[23] Leiva C and Plyushchay M S 2003 *J. High Energy Phys.* JHEP10(2003)069
Plyushchay M 2004 *J. Phys. A: Math. Gen.* **37** 10375
[24] Mielnik B and Rosas-Ortiz O 2004 *J. Phys. A: Math. Gen.* **37** 10007
[25] Ioffe M V and Nishnianidze D N 2004 *Phys. Lett.* A **327** 425

[26] González-López A and Tanaka T 2004 *Phys. Lett.* B **586** 117
[27] Sukumar C V 2005 *AIP Conf. Proc.* **744** 166
[28] Fernández D J and Fernández-García N 2005 *AIP Conf. Proc.* **744** 236
[29] Fernández D J, Hussin V and Nieto L M 1994 *J. Phys. A: Math. Gen.* **27** 3547
Fernández D J, Nieto L M and Rosas-Ortiz O 1995 *J. Phys. A: Math. Gen.* **28** 2693
Rosas-Ortiz J O 1996 *J. Phys. A: Math. Gen.* **29** 3281
[30] Kumar M S and Khare A 1996 *Phys. Lett.* A **217** 73
[31] Bagrov V G and Samsonov B F 1996 *J. Phys. A: Math. Gen.* **29** 1011
[32] Aizawa N and Sato H T 1997 *Prog. Theor. Phys.* **98** 707
[33] Seshadri S, Balakrishnan V and Lakshmibala S 1998 *J. Math. Phys.* **39** 838
[34] Fernández D J and Hussin V 1999 *J. Phys. A: Math. Gen.* **32** 3603
[35] Fukui T and Aizawa N 1993 *Phys. Lett.* A **180** 308
[36] Samsonov B F 1998 *JETP* **87** 1046
[37] Daoud M and Hussin V 2002 *J. Phys. A: Math. Gen.* **35** 7381
[38] Shreecharan T, Panigrahi P K and Banerji J 2004 *Phys. Rev.* A **69** 012102
Roy U, Banerji J and Panigrahi P K 2005 *J. Phys. A: Math. Gen.* **38** 9115
[39] Hubbard J 1964 *Proc. R. Soc.* A **277** 237
[40] Förster D 1989 *Phys. Rev. Lett.* **63** 2140
[41] Coleman P, Pépin C and Hopkinson J 2001 *Phys. Rev.* B **63** 140411
[42] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[43] Penson K A and Solomon A I 1999 *J. Math. Phys.* **40** 2354
Sixdeniers J M, Penson K A and Solomon A I 1999 *J. Phys. A: Math. Gen.* **32** 7543
Klauder J R, Penson K A and Sixdeniers J M 2001 *Phys. Rev.* A **64** 013817
[44] Sixdeniers J M and Penson K A 2000 *J. Phys. A: Math. Gen.* **33** 2907
Sixdeniers J M and Penson K A 2001 *J. Phys. A: Math. Gen.* **34** 2859
[45] Mielnik B, Nieto L M and Rosas-Ortiz O 2000 *Phys. Lett.* A **269** 70
[46] Fernández D J and Salinas-Hernández E 2003 *J. Phys. A: Math. Gen.* **36** 2537
Fernández D J and Salinas-Hernández E 2005 *Phys. Lett.* A **338** 13
[47] Mielnik B 1984 *J. Math. Phys.* **25** 3387
[48] Nieto M M 1978 *Phys. Rev.* A **17** 1273